Math 259A Lecture 13 Notes

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1 Cyclic and Separating Vectors, and The Extension of The Gelfand Transform

1.1 Cyclic and separating vectors

Definition 1.1. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **cyclic vector** of M if $\overline{M\xi} = H$.

Definition 1.2. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. $\xi \in H$ is a **separating vector** of M if when $x \in M$ satisfies $x\xi = 0$, $x = 0$.

Proposition 1.1. *Let $M = \mathcal{A}$ be an abelian von Neumann algebra. If ξ is cyclic, it is separating.*

Proof. If ξ is cyclic and $x\xi = 0$, then $\mathcal{A}(x\xi) = 0$. So $x\overline{\mathcal{A}\xi} = 0$. So $x = 0$. $\square$

Definition 1.3. If $\{p_i\}$ are projections in M with $p_i p_j = 0$, then we define $\sum_i p_i := \bigvee_i p_i$.

Lemma 1.1. *If $\mathcal{A} \subseteq \mathcal{B}(H)$ is an abelian von Neumann algebra, then $\mathcal{A}$ has a separating vector.*

Proof. Let $\{\xi_i\}_{i \in I}$ be a maximal family of unit vectors such that $[A\xi_i]$ is mutually orthogonal. Then $\sum_i [A\xi_i] = 1$. To see why, suppose not. Then for $1 - \sum_i [A\xi_i] \neq 0$, take ξ_0 be a unit vector in the range of $1 - \sum_i [A\xi_i]$. Then for any fixed i, $\langle x\xi_0, y\xi_i \rangle = \langle \xi_0, x^* y \xi_i \rangle = 0$.

This implies that $\{\xi_i\}$ is countable, so let $\xi = \sum_{n \geq 1} 2^{-n} \xi_n$. We claim that if $x \in \mathcal{A}$ and $x\xi = 0$, then $x = 0$. Indeed, if $x\xi = 0$, then $[A\xi_n] x\xi = 0$, so $0 = x[A\xi_n](\xi) = 2^{-n}\xi_n$. This shows that $\xi_n = 0$ for all n, so $x[A\xi_n] = 0$ for all n. So $xH = 0$, making $x = 0$. $\square$

Corollary 1.1. *Let H be separable, and let $\mathcal{A} \subseteq \mathcal{B}(H)$ be an abelian von Neumann algebra with $\xi \in H$ separating for $\mathcal{A}$. Let $p = p_{H_0} = [\mathcal{A}\xi]$. Then the map $\mathcal{A} \mapsto \mathcal{B}(H_0)$ given by $x \mapsto xp$ is a 1 to 1 *-algebra morphism which is SO-SO[1] continuous (with SO-SO continuous inverse).*

Remark 1.1. We can also say this is WO-WO continuous.

[1] This doesn't mean that it's only sort of continuous. But I know you had the thought.

1.2 Extension of the Gelfand transform

Theorem 1.1. *Let $T \in \mathcal{B}(H)$ be a normal operator, let $\mathcal{A}_T = \{T, T^*\}''$ be the von Neumann algebra generated by T. Assume $\mathcal{A}_T$ has a cyclic vector $\xi \in H$ with $\|\xi\| = 1$. Then there exist a positive, regular Borel measure μ on $X = \mathrm{Spec}(T) \subseteq \mathbb{C}$ of support X, a unitary $U : H \to L^2(X, \mu)$, and an isometric *-morphism $\Phi : \mathcal{A}_T \to \mathcal{B}(L^2(X, \mu))$ implemented spactially by U; i.e. $\Phi(x) = UxU^{-1} \in \mathcal{B}(L^2(X, \mu))$. Moreover, Φ has range $\{M_f : f \in L^\infty(X, \mu)\}$, which is maximal abelian in $\mathcal{B}(L^2(X, \mu))$, and, when restricted to the C^*-algebra generated by T, T^*, is the Gelfand transform. In particular, $\Phi(T^n) = M_{z^n}$, $\Phi((T^*)^n) = M_{\overline{z}^n}$. The measure μ is given by*

$$\int_X f\, d\mu = \langle f(T)\xi, \xi \rangle .$$

Uniqueness: If μ_1 is a positive, regular Borel measure on $\mathbb{C}$ with $\mathrm{supp}(\mu_1) = \mathrm{Spec}(T)$ and $\Phi_1 : \mathcal{A}_T \to L^\infty(X, \mu_1)$ extends Γ, then $\mu \sim \mu_1$ and $\Phi_1 = \Phi$.

Proof. Read the Douglas textbook for the proof. □

Now if $T \in \mathcal{B}(H)$ is an arbitrary normal operator, what is its **Borel/L^∞ calculus**? Take a separating $\xi \in H$ for $\mathcal{A}_T = \{T, T^*\}''$. Then $\mathcal{A}_T \mapsto \mathcal{A}_T p \in \mathcal{B}([\mathcal{A}_T \xi])$ identifies $(\mathcal{A}_T, \langle \cdot, \xi, \xi \rangle) \to (L^\infty(\mathrm{Spec}(T)s, \mu), \mu)$.

1.3 Projection geometry

Let $\mathcal{P}(M)$ denote the projections in the von Neumann algebra M.

Definition 1.4. If $e, f \in \mathcal{P}(M)$, then $e \sim f$ if there exists a partial isometry $v \in M$ with $\ell(v) = e$ and $r(v) = f$; i.e. $vv^* = e$ and $v^*v = f$.

Theorem 1.2. *If $x \in M$, then $\ell(x) \sim r(x)$.*

Proof. This is by the polar decomposition of x. □

Theorem 1.3 (Paralellogram rule)**.** *If $e, f \in M$, then $(e \vee f - f) \sim (e - e \wedge f)$.*

Proof. Use the fact that $e \vee f - f = \ell(e(1 - f))$, and $e - e \wedge f = r(e(1 - f))$. □

Theorem 1.4 (Cantor-Bernstein)**.** *If $e \prec f$ and $f \prec e$, then $e \sim f$.*